

Massive spin-2 in the Fradkin-Vasiliev formalism

I. Partially massless case

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Abstract

We apply Fradkin-Vasiliev formalism to construction of non-trivial cubic interaction vertices for massive spin-2 particles. In this first paper as a relatively simple but instructive example we consider self-interaction and gravitational interaction of partially massless spin-2.

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Introduction

Last years there were lot of activities in the investigation of consistent cubic interaction vertices for higher spin fields. Such investigations are very important steps in the search for consistent higher spin theories and, in particular, provide information on the possible gauge symmetry algebras behind such models. Till now most of the results were devoted to cubic vertices for massless higher spin fields and now we have rather good understanding of their properties. Moreover, the results obtained by different groups and different methods are perfectly consistent, see e.g. [1]-[10].

At the same time investigations of cubic vertices containing massive higher spin fields are not so numerous, the most important one being classification of cubic vertices in flat Minkowski space by Metsaev [11, 12, 13], while there also exist a number of concrete examples e.g. [14]-[24].

One of the approaches that turned out to be very effective for investigation of massless higher spin fields interactions is the Fradkin-Vasiliev formalism [25, 26] (see also [27, 28, 29, 30]). Let us briefly remind how this formalism works. The basis for the whole construction is the frame-like formalism [31, 32, 33], where massless higher spin particle is described by a set of (physical, auxiliary and extra) one forms that we will collectively denote as Φ here. As far as the free theory is concerned, three most important facts are:

- each field has its own gauge transformation

$$\delta\Phi \sim D\xi \oplus e\xi$$

where D is $(A)dS$ covariant derivative, while e is background $(A)dS$ frame;

- gauge invariant two-form (curvature) can be constructed for each field

$$R \sim D \wedge \Phi \oplus e \wedge \Phi$$

- free Lagrangian can be rewritten in an explicitly gauge invariant form

$$\mathcal{L}_0 \sim \sum R \wedge R$$

Using these ingredients cubic interaction vertices can be constructed by the following straightforward steps.

- Take the most general quadratic deformation for curvatures

$$\hat{R} = R \oplus \Phi \wedge \Phi$$

as a result these new deformed curvatures ceased to be invariant

$$\delta\hat{R} \sim \Phi \wedge D\xi \oplus e \wedge \Phi\xi$$

- Introduce corrections to gauge transformations

$$\delta\Phi \sim \Phi\xi$$

in such a way that

$$\delta\hat{R} \sim D \wedge \Phi\xi \oplus e \wedge \Phi\xi$$

- Adjust coefficients so that deformed curvatures transform covariantly

$$\delta \hat{R} \sim R\xi$$

- At last, consider the Lagrangian in the form

$$\mathcal{L} \sim \sum \hat{R} \wedge \hat{R} \oplus \sum R \wedge R \wedge \Phi$$

where the first part is just the sum of free Lagrangians with initial curvatures replaced by the deformed ones, while the second part contains all possible abelian vertices. By construction and due to Bianchi identities all variations for such Lagrangian take the form

$$\delta \mathcal{L} \sim R \wedge R\xi$$

reducing the problem to the set of algebraic equations. Moreover, Vasiliev has shown [34] that for the three massless fields with arbitrary spins s_1 , s_2 and s_3 all non-trivial cubic vertices having up to $s_1 + s_2 + s_3 - 2$ derivatives can be constructed in this way.

As we have seen two main ingredients of this approach are frame-like formalism and gauge invariance. But frame-like gauge invariant description exists for massive higher spin fields as well [35]-[39]. Thus it seems natural to extend Fradkin-Vasiliev formalism to the cases where both massive and (partially) massless fields are present. Such approach has already been successfully applied to the investigation of gravitational and electromagnetic interactions for simplest massive mixed symmetry field [40, 41]. Now we are going to apply this approach to the construction of cubic vertices for massive spin-2 particles¹. In this first paper we restrict ourselves with relatively simple but instructive case of partially massless spin-2 field [42, 43, 44, 45] leaving general massive case for the second part.

The paper is organized as follows. In Section 1 we illustrate general approach on the simplest but physically important example of massless spin-2 field in $(A)dS$ space. Namely, we consider both self-interaction as well as gravitational interaction vertices for such field. The main section 2 is devoted to the partially massless spin-2 case. First of all in subsection 2.1 we provide all necessary kinematic formulas. Then in subsections 2.2 and 2.3 we consider self-interaction and gravitational interaction correspondingly. Due to the presence of zero forms as well as a number of identities making different terms equivalent on-shell, an analysis turns out to be more complicated than in the purely massless case. Thus as an independent check for the number of non-equivalent cubic vertices (as well as very instructive comparison) in Appendix we reconsider the same vertices in a straightforward constructive approach.

Notations and conventions We work in $(A)dS$ space with dimension $d \geq 4$ with (non-dynamical) background frame $e_\mu{}^a$ and $(A)dS$ covariant derivative D_μ normalized so that

$$[D_\mu, D_\nu]\xi^a = -\kappa(e_\mu{}^a \xi_\nu - e_\nu{}^a \xi_\mu), \quad \kappa = \frac{2\Lambda}{(d-1)(d-2)}$$

Here Greek letters are used for the world indices, while Latin letters denote local ones. As it common for the frame-like formalism, all terms in the Lagrangians will be completely antisymmetric on world indices and we will heavily use notations like

$$\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \} = e^\mu{}_a e^\nu{}_b - e^\mu{}_b e^\nu{}_a$$

¹Let us stress that we consider massive spin-2 as a simple representative of massive higher spin fields, and not as massive graviton.

1 Massless case

1.1 Kinematics

In the frame-like formalism free Lagrangian for massless spin 2 field in $(A)dS$ background has the form:

$$\mathcal{L}_0 = \frac{1}{2} \{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \} \omega_\mu^{ac} \omega_\nu^{bc} - \frac{1}{2} \{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \} \omega_\mu^{ab} D_\nu h_\alpha^c - \frac{(d-2)\kappa}{2} \{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \} h_\mu^a h_\nu^b \quad (1)$$

This Lagrangian is invariant under the following gauge transformations:

$$\delta_0 \omega_\mu^{ab} = D_\mu \hat{\eta}^{ab} + \kappa e_\mu^{[a} \hat{\xi}^{b]}, \quad \delta_0 h_\mu^a = D_\mu \hat{\xi}^a + \hat{\eta}_\mu^a \quad (2)$$

It is easy to construct two gauge invariant objects (linearized curvature and torsion):

$$\begin{aligned} R_{\mu\nu}^{ab} &= D_{[\mu} \omega_{\nu]}^{ab} + \kappa e_{[\mu}^{[a} h_{\nu]}^{b]} \\ T_{\mu\nu}^a &= D_{[\mu} h_{\nu]}^a - \omega_{[\mu, \nu]}^a \end{aligned} \quad (3)$$

Differential identities for them look like:

$$D_{[\mu} R_{\nu\alpha]}^{ab} = -\kappa e_{[\mu}^{[a} T_{\nu\alpha]}^{b]}, \quad D_{[\mu} T_{\nu\alpha]}^a = -R_{[\mu\nu, \alpha]}^a \quad (4)$$

Note that on mass shell for auxiliary field ω_μ^{ab} we have

$$T_{\mu\nu}^a \approx 0 \quad \Rightarrow \quad R_{[\mu\nu, \alpha]}^a \approx 0, \quad D_{[\mu} R_{\nu\alpha]}^{ab} \approx 0$$

The free Lagrangian can be rewritten in the explicitly gauge invariant form:

$$\mathcal{L}_0 = a_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} R_{\mu\nu}^{ab} R_{\alpha\beta}^{cd}, \quad a_0 = -\frac{1}{32(d-3)\kappa} \quad (5)$$

1.2 Self-interaction

The most general quadratic deformations for curvatures have the form:

$$\begin{aligned} \Delta R_{\mu\nu}^{ab} &= b_0 \omega_{[\mu}^{ca} \omega_{\nu]}^{bc} + b_1 h_{[\mu}^a h_{\nu]}^b \\ \Delta T_{\mu\nu}^a &= b_2 \omega_{[\mu}^{ab} h_{\nu]}^b \end{aligned} \quad (6)$$

If we require that deformed curvatures transform covariantly we have to put:

$$b_1 = \kappa b_0, \quad b_2 = b_0$$

In this case corrections to the initial gauge transformations look like

$$\begin{aligned} \delta_1 \omega_\mu^{ab} &= b_0 [\omega_\mu^{c[a} \hat{\eta}^{b]c} + \kappa h_\mu^{[a} \hat{\xi}^{b]}] \\ \delta_1 h_\mu^a &= b_0 [-\hat{\eta}^{ab} h_\mu^b + \omega_\mu^{ab} \hat{\xi}^b] \end{aligned} \quad (7)$$

while deformed curvatures transform as follows:

$$\begin{aligned} \delta \hat{R}_{\mu\nu}^{ab} &= b_0 R_{\mu\nu}^{c[a} \hat{\eta}^{b]c} + \kappa b_0 T_{\mu\nu}^{[a} \hat{\xi}^{b]} \\ \delta \hat{T}_{\mu\nu}^a &= -b_0 \hat{\eta}^{ab} T_{\mu\nu}^b + b_0 R_{\mu\nu}^{ab} \hat{\xi}^b \end{aligned} \quad (8)$$

Let us consider the following Lagrangian

$$\mathcal{L} = a_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \hat{R}_{\mu\nu}{}^{ab} \hat{R}_{\alpha\beta}{}^{cd} + c_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} R_{\mu\nu}{}^{ab} R_{\alpha\beta}{}^{cd} h_{\gamma}{}^e \quad (9)$$

where the first term is just the free Lagrangian where initial curvature is replaced by the deformed one, while the second term is an abelian vertex. Using identities given above it is easy to check that both terms are gauge invariant on-shell. This Lagrangian gives the following cubic vertex (here and in what follows the second index denotes the number of derivatives in the vertex²):

$$\mathcal{L}_1 = \mathcal{L}_{14} + \mathcal{L}_{12} + \mathcal{L}_{10}$$

$$\begin{aligned} \mathcal{L}_{14} &= -8a_0b_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_{\mu}\omega_{\nu}{}^{ab}\omega_{\alpha}{}^{ce}\omega_{\beta}{}^{de} + 4c_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} D_{\mu}\omega_{\nu}{}^{ab}D_{\alpha}\omega_{\beta}{}^{cd}h_{\gamma}{}^e \\ \mathcal{L}_{12} &= 8\kappa(a_0b_0 + 2c_0(d-4)) \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_{\mu}\omega_{\nu}{}^{ab}h_{\alpha}{}^ch_{\beta}{}^d \\ &\quad - 16a_0b_0(d-3)\kappa \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} h_{\mu}{}^a\omega_{\nu}{}^{bd}\omega_{\alpha}{}^{cd} \\ \mathcal{L}_{10} &= 16(d-3)\kappa^2(a_0b_0 + c_0(d-4)) \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} h_{\mu}{}^ah_{\nu}{}^bh_{\alpha}{}^c \end{aligned}$$

Thus we have two independent vertices with terms up to four derivatives³. But on-shell we have

$$\begin{aligned} 4c_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} D_{\mu}\omega_{\nu}{}^{ab}D_{\alpha}\omega_{\beta}{}^{cd}h_{\gamma}{}^e &\approx 12c_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_{\mu}\omega_{\nu}{}^{ab}\omega_{\alpha}{}^{ce}\omega_{\beta}{}^{de} - \\ &- 8c_0(d-4)\kappa \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} [2\omega_{\mu}{}^{ad}\omega_{\nu}{}^{bd}h_{\alpha}{}^c + \omega_{\mu}{}^{ab}\omega_{\nu}{}^{cd}h_{\alpha}{}^d] \end{aligned}$$

Thus if we put

$$c_0 = \frac{2a_0b_0}{3}$$

all four derivative terms cancel on-shell leaving us with the vertex containing no more than two derivatives:

$$\mathcal{L}_1 = \frac{b_0}{2} [\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \omega_{\mu}{}^{ad}\omega_{\nu}{}^{bd}h_{\alpha}{}^c - \frac{1}{2} \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_{\mu}\omega_{\nu}{}^{ab}h_{\alpha}{}^ch_{\beta}{}^d - \frac{(2d-5)\kappa}{3} \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} h_{\mu}{}^ah_{\nu}{}^bh_{\alpha}{}^c] \quad (10)$$

1.3 Gravitational interaction

For the second spin-2 we will use notations $(\Omega_{\mu}{}^{ab}, f_{\mu}{}^a)$, (η^{ab}, ξ^a) and $(\mathcal{F}_{\mu\nu}{}^{ab}, \mathcal{T}_{\mu\nu}{}^a)$ for fields, gauge parameters and gauge invariant curvatures correspondingly.

Let us consider gravitational interactions for this second spin-2. Similarly to the previous case for the deformations of gravitational curvatures we obtain

$$\begin{aligned} \Delta R_{\mu\nu}{}^{ab} &= b_0[\Omega_{[\mu}{}^{ca}\Omega_{\nu]}{}^{bc} + \kappa f_{[\mu}{}^af_{\nu]}{}^b] \\ \Delta T_{\mu\nu}{}^a &= b_0\Omega_{[\mu}{}^{ab}f_{\nu]}{}^b \end{aligned} \quad (11)$$

²Calculating the number of derivatives we take into account that auxiliary field $\omega_{\mu}{}^{ab}$ is equivalent to the first derivative of physical field.

³We are working in the linear approximation so for any two solutions their arbitrary linear combination is also a solution. Thus the number of independent solutions is just the number of free parameters.

while deformed curvatures will transform as follows:

$$\begin{aligned}\delta \hat{R}_{\mu\nu}{}^{ab} &= b_0 \mathcal{F}_{\mu\nu}{}^{c[a} \eta^{b]c} + \kappa b_0 \mathcal{T}_{\mu\nu}{}^{[a} \xi^{b]} \\ \delta \hat{T}_{\mu\nu}{}^a &= -b_0 \eta^{ab} \mathcal{T}_{\mu\nu}{}^b + b_0 \mathcal{F}_{\mu\nu}{}^{ab} \xi^b\end{aligned}\quad (12)$$

As for the deformations for the second spin-2 curvatures they correspond to standard minimal substitution rules for gravitational interactions:

$$\begin{aligned}\Delta \mathcal{F}_{\mu\nu}{}^{ab} &= b_1 [\Omega_{[\mu}{}^{c[a} \omega_{\nu]}{}^{b]c} + \kappa f_{[\mu}{}^{[a} h_{\nu]}{}^{b]}] \\ \Delta \mathcal{T}_{\mu\nu}{}^a &= b_1 [\Omega_{[\mu}{}^{ab} h_{\nu]}{}^b + \omega_{[\mu}{}^{ab} f_{\nu]}{}^b]\end{aligned}\quad (13)$$

while transformation rules for them look like:

$$\begin{aligned}\delta \hat{\mathcal{F}}_{\mu\nu}{}^{ab} &= b_1 [-\hat{\eta}^{c[a} \mathcal{F}_{\mu\nu}{}^{b]c} + \kappa \mathcal{T}_{\mu\nu}{}^{[a} \hat{\xi}^{b]} - \eta^{c[a} R_{\mu\nu}{}^{b]c} + \kappa T_{\mu\nu}{}^{[a} \xi^{b]}] \\ \delta \hat{\mathcal{T}}_{\mu\nu}{}^a &= b_1 [-\hat{\eta}^{ab} \mathcal{T}_{\mu\nu}{}^b + \mathcal{F}_{\mu\nu}{}^{ab} \hat{\xi}^b - \eta^{ab} T_{\mu\nu}{}^b + R_{\mu\nu}{}^{ab} \xi^b]\end{aligned}\quad (14)$$

Note that at this stage two parameters b_0 and b_1 are independent and it may seem that it contradicts with the universality of gravitational interactions. The reason is that covariance of deformed curvatures guarantees that equation of motion for the theory we are trying to construct will be gauge invariant but it does not guarantee that these equations will be Lagrangean. Thus if we put these deformed curvatures into the Lagrangian and require that this Lagrangian be invariant we have to expect that parameters b_0 and b_1 will be related. As we will see right now it turns out to be the case.

Let us consider the following Lagrangian:

$$\mathcal{L} = a_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [\hat{\mathcal{F}}_{\mu\nu}{}^{ab} \hat{\mathcal{F}}_{\alpha\beta}{}^{cd} + \hat{R}_{\mu\nu}{}^{ab} \hat{R}_{\alpha\beta}{}^{cd}] + c_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ab} \mathcal{F}_{\alpha\beta}{}^{cd} h_{\gamma}{}^e \quad (15)$$

where the first two terms are just the sum of free Lagrangians with initial curvatures replaced by the deformed ones, while the last term is an abelian vertex. Note that there is one more abelian vertex

$$\Delta \mathcal{L} = c_2 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ab} R_{\alpha\beta}{}^{cd} f_{\gamma}{}^e$$

but (as we have explicitly checked) this vertex completely equivalent on-shell to the one with coefficient c_1 so we will not introduce it here. Let us take transformations of curvatures that do not vanish on-shell:

$$\begin{aligned}\delta \hat{\mathcal{F}}_{\mu\nu}{}^{ab} &= -b_1 [\hat{\eta}^{c[a} \mathcal{F}_{\mu\nu}{}^{b]c} + \eta^{c[a} R_{\mu\nu}{}^{b]c}] \\ \delta \hat{R}_{\mu\nu}{}^{ab} &= -b_0 \eta^{c[a} \mathcal{F}_{\mu\nu}{}^{b]c}\end{aligned}\quad (16)$$

Variations under the $\hat{\eta}^{ab}$ transformations trivially vanish on-shell, so let us consider the ones for the η^{ab} transformations:

$$-4a_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [b_1 \mathcal{F}_{\mu\nu}{}^{ab} R_{\alpha\beta}{}^{ce} \eta^{de} + b_0 \mathcal{F}_{\mu\nu}{}^{ae} R_{\alpha\beta}{}^{bc} \eta^{de}]$$

But on-shell we have two identities

$$0 \approx \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu,\alpha}{}^a R_{\alpha\beta}{}^{bc} \eta^{de} = 2 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ae} [-R_{\alpha\beta}{}^{be} \eta^{cd} - R_{\alpha\beta}{}^{bc} \eta^{de}]$$

$$0 \approx \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ab} R_{\alpha\beta,\gamma}{}^c \eta^{de} = 2 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [\mathcal{F}_{\mu\nu}{}^{ae} R_{\alpha\beta}{}^{be} \eta^{cd} - \mathcal{F}_{\mu\nu}{}^{ab} R_{\alpha\beta}{}^{ce} \eta^{de}]$$

and their combination gives us

$$\left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [\mathcal{F}_{\mu\nu}{}^{ab} R_{\alpha\beta}{}^{ce} + \mathcal{F}_{\mu\nu}{}^{ae} R_{\alpha\beta}{}^{bc}] \eta^{de} \approx 0 \quad (17)$$

Thus we have to put (as expected)

$$b_1 = b_0 \quad (18)$$

So, as in the previous case, we have two independent vertices with free parameters b_0 and c_1 and terms containing up to four derivatives. Let us extract all the terms for the cubic vertex:

$$\mathcal{L}_1 = \mathcal{L}_{14} + \mathcal{L}_{12} + \mathcal{L}_{10}$$

$$\begin{aligned} \mathcal{L}_{14} &= -8a_0 b_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [2D_\mu \Omega_\nu{}^{ab} \Omega_\alpha{}^{ce} \omega_\beta{}^{de} + D_\mu \omega_\nu{}^{ab} \Omega_\alpha{}^{ce} \Omega_\beta{}^{de}] \\ &\quad + 4c_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} D_\mu \Omega_\nu{}^{ab} D_\alpha \Omega_\beta{}^{cd} h_\gamma{}^e \\ \mathcal{L}_{12} &= -8a_0 b_0 \kappa (d-3) \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} [\Omega_\mu{}^{ad} \Omega_\nu{}^{bd} + 2\Omega_\mu{}^{ad} \omega_\nu{}^{bd} f_\alpha{}^c] \\ &\quad + 8\kappa \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [(2a_0 b_0 + 2(d-4)c_1) D_\mu \Omega_\nu{}^{ab} f_\alpha{}^c h_\beta{}^d + a_0 b_0 D_\mu \omega_\nu{}^{ab} f_\alpha{}^c f_\beta{}^d] \\ \mathcal{L}_{10} &= 16\kappa^2 (d-3) (3a_0 b_0 + (d-4)c_1) \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} f_\mu{}^a f_\nu{}^b h_\alpha{}^c \end{aligned}$$

But on the auxiliary fields $\omega_\mu{}^{ab}$ and $\Omega_\mu{}^{ab}$ mass shell we have

$$\begin{aligned} 4c_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} D_\mu \Omega_\nu{}^{ab} D_\alpha \Omega_\beta{}^{cd} h_\gamma{}^e &\approx 4c_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [2D_\mu \Omega_\nu{}^{ab} \Omega_\alpha{}^{ce} \omega_\beta{}^{de} + D_\mu \omega_\nu{}^{ab} \Omega_\alpha{}^{ce} \Omega_\beta{}^{de}] \\ &\quad - 4\kappa (d-4) c_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [2D_\mu \Omega_\nu{}^{ab} f_\alpha{}^c h_\beta{}^d - D_\mu \omega_\nu{}^{ab} f_\alpha{}^c f_\beta{}^d] \end{aligned}$$

thus for $c_1 = 2a_0 b_0$ all four derivative terms vanish leaving us with the vertex containing no more than two derivatives:

$$\begin{aligned} \mathcal{L}_1 &= \frac{b_0}{4} \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} [\Omega_\mu{}^{ad} \Omega_\nu{}^{bd} + 2\Omega_\mu{}^{ad} \omega_\nu{}^{bd} f_\alpha{}^c] \\ &\quad - \frac{b_0}{4} \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [2D_\mu \Omega_\nu{}^{ab} f_\alpha{}^c h_\beta{}^d + D_\mu \omega_\nu{}^{ab} f_\alpha{}^c f_\beta{}^d] \\ &\quad - \frac{(2d-5)\kappa b_0}{2} \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} f_\mu{}^a f_\nu{}^b h_\alpha{}^c \end{aligned} \quad (19)$$

2 Partially massless case

2.1 Kinematics

In the frame-like formalism gauge invariant description for the partially massless spin-2 particle [35, 39] requires two pairs of (auxiliary and physical) fields: $(\Omega_\mu{}^{ab}, f_\mu{}^a)$ and (B^{ab}, B_μ) . Free Lagrangian for such particle has the form:

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2} \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \Omega_\mu{}^{ac} \Omega_\nu{}^{bc} - \frac{1}{2} \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \Omega_\mu{}^{ab} D_\nu f_\alpha{}^c + \frac{1}{2} B_{ab}{}^2 - \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} B^{ab} D_\mu B_\nu \\ &\quad + m[\left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \omega_\mu{}^{ab} B_\nu + e^\mu{}_a B^{ab} f_\mu{}^b] \end{aligned} \quad (20)$$

where $m^2 = (d-2)\kappa$. This Lagrangian is invariant under the following gauge transformations:

$$\begin{aligned}\delta_0 \Omega_\mu^{ab} &= D_\mu \eta^{ab}, & \delta_0 f_\mu^a &= D_\mu \xi^a + \eta_\mu^a + \frac{2m}{(d-2)} e_\mu^a \xi \\ \delta_0 B^{ab} &= -m \eta^{ab}, & \delta_0 B_\mu &= D_\mu \xi + \frac{m}{2} \xi_\mu\end{aligned}\tag{21}$$

Correspondingly, we have four gauge invariant objects (curvatures):

$$\begin{aligned}\mathcal{F}_{\mu\nu}^{ab} &= D_{[\mu} \Omega_{\nu]}^{ab} - \frac{m}{(d-2)} e_{[\mu}^{[a} B_{\nu]}^{b]} \\ \mathcal{T}_{\mu\nu}^a &= D_{[\mu} f_{\nu]}^a - \Omega_{[\mu, \nu]}^a + \frac{2m}{(d-2)} e_{[\mu}^a B_{\nu]} \\ \mathcal{B}_\mu^{ab} &= D_\mu B^{ab} + m \Omega_\mu^{ab} \\ \mathcal{B}_{\mu\nu} &= D_{[\mu} B_{\nu]} - B_{\mu\nu} - \frac{m}{2} f_{[\mu, \nu]}\end{aligned}\tag{22}$$

They satisfy the following differential identities:

$$\begin{aligned}D_{[\mu} \mathcal{F}_{\nu\alpha]}^{ab} &= \frac{m}{(d-2)} e_{[\mu}^{[a} \mathcal{B}_{\nu, \alpha]}^{b]} \\ D_{[\mu} \mathcal{T}_{\nu\alpha]}^a &= -\mathcal{F}_{[\mu\nu, \alpha]}^a - \frac{2m}{(d-2)} e_{[\mu}^a \mathcal{B}_{\nu\alpha]} \\ D_{[\mu} \mathcal{B}_{\nu]}^{ab} &= m \mathcal{F}_{\mu\nu}^{ab} \\ D_{[\mu} \mathcal{B}_{\nu\alpha]} &= -\mathcal{B}_{[\mu, \nu\alpha]} - \frac{m}{2} \mathcal{T}_{[\mu\nu, \alpha]}\end{aligned}\tag{23}$$

Note that on mass shell for auxiliary fields Ω_μ^{ab} and B^{ab} we have

$$\mathcal{T}_{\mu\nu}^a \approx 0, \quad \mathcal{B}_{\mu\nu} \approx 0 \quad \Rightarrow \quad \mathcal{F}_{[\mu\nu, \alpha]}^a \approx 0, \quad \mathcal{B}_{[\mu, \nu\alpha]} \approx 0$$

Using these curvatures the free Lagrangian can be rewritten in an explicitly gauge invariant form

$$\mathcal{L}_0 = a_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ab} \mathcal{F}_{\alpha\beta}^{cd} + a_2 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \mathcal{B}_\mu^{ac} \mathcal{B}_\nu^{bc} + a_3 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{B}_\mu^{ab} \mathcal{T}_{\nu\alpha}^c\tag{24}$$

where

$$\frac{16(d-3)}{(d-2)} a_1 - a_2 = \frac{1}{2m^2}, \quad a_3 = -\frac{1}{4m}$$

The ambiguity with coefficients is related with the identity

$$\begin{aligned}0 &= \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_\mu [\mathcal{F}_{\nu\alpha}^{ab} \mathcal{B}_\beta^{cd}] = \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} [\mathcal{F}_{\mu\nu}^{ab} D_\alpha \mathcal{B}_\beta^{cd} + D_\mu \mathcal{F}_{\nu\alpha}^{ab} \mathcal{B}_\beta^{cd}] \\ &= \frac{m}{2} \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ab} \mathcal{F}_{\alpha\beta}^{cd} + \frac{8m(d-3)}{(d-2)} \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \mathcal{B}_\mu^{ac} \mathcal{B}_\nu^{bc}\end{aligned}$$

In what follows we will use the convenient choice

$$a_1 = -\frac{(d-2)}{32(d-3)m^2}, \quad a_2 = -\frac{1}{m^2}, \quad a_3 = -\frac{1}{4m}\tag{25}$$

2.2 Self-interaction

Following general procedure we begin with the most general quadratic deformations for all four curvatures:

$$\begin{aligned}
\Delta \mathcal{F}_{\mu\nu}{}^{ab} &= d_1 \Omega_{[\mu}{}^{c[a} \Omega_{\nu]}{}^{b]c} + d_2 B_{[\mu}{}^{[a} B_{\nu]}{}^{b]} + d_3 B^{ab} B_{\mu\nu} + d_4 e_{[\mu}{}^{[a} B^{b]c} B_{\nu]}{}^c + d_5 e_{[\mu}{}^a e_{\nu]}{}^b B^{cd} B^{cd} \\
&\quad + d_6 \Omega_{[\mu}{}^{ab} B_{\nu]}{}^b + d_7 B_{[\mu}{}^{[a} f_{\nu]}{}^{b]} + d_8 B^{ab} f_{[\mu,\nu]} + d_9 e_{[\mu}{}^{[a} B^{b]c} f_{\nu]}{}^c + d_{10} f_{[\mu}{}^{[a} f_{\nu]}{}^{b]} \\
\Delta \mathcal{T}_{\mu\nu}{}^a &= d_{11} \Omega_{[\mu}{}^{ab} B_{\nu]}{}^b + d_{12} B^{ab} \Omega_{[\mu,\nu]}{}^b + d_{13} e_{[\mu}{}^a \Omega_{\nu]}{}^{bc} B^{bc} \\
&\quad + d_{14} \Omega_{[\mu}{}^{ab} f_{\nu]}{}^b + d_{15} B_{[\mu}{}^a B_{\nu]} + d_{16} f_{[\mu}{}^a B_{\nu]} \\
\Delta \mathcal{B}_{\mu}{}^{ab} &= d_{17} \Omega_{\mu}{}^{c[a} B^{b]c} + d_{18} B^{ab} B_{\mu} \\
\Delta \mathcal{B}_{\mu\nu} &= d_{19} B_{[\mu}{}^a f_{\nu]}{}^a
\end{aligned}$$

Usual requirement that deformed curvatures transform covariantly gives solution with five arbitrary parameters. However, due to the presence of zero form B^{ab} there are four possible field redefinitions (their explicit action can be seen in the Appendix):

$$\Omega_{\mu}{}^{ab} \Rightarrow \Omega_{\mu}{}^{ab} + \kappa_1 B^{ab} B_{\mu}, \quad f_{\mu}{}^a \Rightarrow f_{\mu}{}^a + \kappa_2 B^{ab} f_{\mu}{}^b + \kappa_3 B^{ab} B_{\mu}{}^b + \kappa_4 e_{\mu}{}^a B^{bc} B^{bc}$$

Using this freedom we can bring the deformations into the form

$$\begin{aligned}
\Delta \mathcal{F}_{\mu\nu}{}^{ab} &= d_1 [\Omega_{[\mu}{}^{c[a} \Omega_{\nu]}{}^{b]c} + \frac{1}{(d-2)} (B_{[\mu}{}^{[a} B_{\nu]}{}^{b]} - e_{[\mu}{}^{[a} B^{b]c} B_{\nu]}{}^c)] + \\
&\quad + d_6 [\Omega_{[\mu}{}^{ab} B_{\nu]}{}^b - \frac{1}{m} B^{ab} B_{\mu\nu} - \frac{1}{2} B^{ab} f_{[\mu,\nu]}] \\
\Delta \mathcal{T}_{\mu\nu}{}^a &= 2d_1 \Omega_{[\mu}{}^{ab} f_{\nu]}{}^b \\
\Delta \mathcal{B}_{\mu}{}^{ab} &= d_1 \Omega_{[\mu}{}^{c[a} B^{b]c} - d_6 B^{ab} B_{\mu} \\
\Delta \mathcal{B}_{\mu\nu} &= -d_1 B_{[\mu}{}^a f_{\nu]}{}^a
\end{aligned} \tag{26}$$

where

$$d_6 = -\frac{4md_1}{(d-2)}$$

This corresponds to the following gauge transformations for the deformed curvatures:

$$\begin{aligned}
\delta \hat{\mathcal{F}}_{\mu\nu}{}^{ab} &= 2d_1 \mathcal{F}_{\mu\nu}{}^{c[a} \eta^{b]c} + \frac{d_6}{2} \mathcal{B}_{[\mu}{}^{ab} \xi_{\nu]} + d_6 \mathcal{F}_{\mu\nu}{}^{ab} \xi \\
\delta \hat{\mathcal{T}}_{\mu\nu}{}^a &= -2d_1 \eta^{ab} \mathcal{T}_{\mu\nu}{}^b + 2d_1 \mathcal{F}_{\mu\nu}{}^{ab} \xi^b \\
\delta \hat{\mathcal{B}}_{\mu}{}^{ab} &= -d_1 \eta^{c[a} \mathcal{B}_{\mu}{}^{b]c} + d_6 \mathcal{B}_{\mu}{}^{ab} \xi \\
\delta \hat{\mathcal{B}}_{\mu\nu} &= -d_1 \mathcal{B}_{[\mu,\nu]}{}^a \xi^a
\end{aligned} \tag{27}$$

Now let us consider the following Lagrangian:

$$\begin{aligned}
\mathcal{L} &= a_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \hat{\mathcal{F}}_{\mu\nu}{}^{ab} \hat{\mathcal{F}}_{\alpha\beta}{}^{cd} + a_2 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \hat{\mathcal{B}}_{\mu}{}^{ac} \hat{\mathcal{B}}_{\nu}{}^{bc} + a_3 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \hat{\mathcal{B}}_{\mu}{}^{ab} \hat{\mathcal{T}}_{\nu\alpha}{}^c \\
&\quad + a_4 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ab} \mathcal{F}_{\alpha\beta}{}^{cd} f_{\gamma}{}^e + a_5 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{B}_{\mu}{}^{ad} \mathcal{B}_{\nu}{}^{bd} f_{\alpha}{}^c + a_6 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ab} \mathcal{B}_{\alpha}{}^{cd} B_{\beta} \tag{28}
\end{aligned}$$

Here the first line is just the free Lagrangian where initial curvatures are replaced by the deformed ones, while the second line contains possible abelian vertices⁴.

Let us require that this Lagrangian be gauge invariant. All η^{ab} variations vanish on-shell. For the ξ variations we obtain

$$\begin{aligned} & (2d_6a_1 + \frac{2(d-4)ma_4}{(d-2)} + \frac{ma_6}{2}) \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ab} \mathcal{F}_{\alpha\beta}{}^{cd} \xi \\ & + (2d_6a_2 + 2ma_5 + \frac{8(d-3)ma_6}{(d-2)}) \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \mathcal{B}_\mu{}^{ac} \mathcal{B}_\nu{}^{bc} \xi = 0 \end{aligned}$$

Thus we have to put

$$2d_6a_1 + \frac{2(d-4)ma_4}{(d-2)} + \frac{ma_6}{2} = 0 \quad (29)$$

$$2d_6a_2 + 2ma_5 + \frac{8(d-3)ma_6}{(d-2)} = 0 \quad (30)$$

For the ξ^a transformations we get

$$\begin{aligned} & (4d_6a_1 + ma_6 + 2d_1a_3) \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ad} \mathcal{B}_\alpha{}^{bc} \xi^d \\ & + (\frac{16(d-4)ma_4}{(d-2)} - ma_5) \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ad} \mathcal{B}_\alpha{}^{bd} \xi^c \end{aligned}$$

and using on-shell identity

$$0 \approx \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu,\alpha}{}^a \mathcal{B}_\beta{}^{bc} \xi^d = \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}{}^{ad} [-2\mathcal{B}_\alpha{}^{bd} \xi^c + \mathcal{B}_\alpha{}^{bc} \xi^d] \quad (31)$$

we obtain

$$8d_6a_1 + 2ma_6 + 4d_1a_3 + \frac{16(d-4)ma_4}{(d-2)} - ma_5 = 0 \quad (32)$$

Thus we obtain three equations which uniquely determines all free coefficients $a_{4,5,6}$ so we have one cubic vertex with terms up to four derivatives. Note that the case $d = 4$ is special because the term with coefficient a_4 is absent. Happily the solution still exists, namely

$$a_5 = -\frac{d_1}{m^2}, \quad a_6 = -\frac{d_1}{2m^2}$$

Moreover, as we have explicitly checked, all cubic terms with four and three derivatives vanish on-shell and we reproduce rather well known two derivative vertex [14, 46, 47]. Note also that the same general results (one four derivative vertex in $d > 4$ and one two derivative vertex in $d = 4$) was obtained also in [22].

⁴Note that in the partially massless case (and in the massive case too) due to peculiarities of gauge transformations the terms in the second line are not gauge invariant separately.

2.3 Gravitational interaction

We begin with the most general quadratic deformations for gravitational curvatures:

$$\begin{aligned}\Delta R_{\mu\nu}{}^{ab} &= b_1 \Omega_{[\mu}{}^c [{}^a \Omega_{\nu]}{}^{b]c} + b_2 B_{[\mu}{}^{[a} B_{\nu]}{}^{b]} + b_3 B_{\mu\nu} B^{ab} + b_4 e_{[\mu}{}^{[a} B^{b]c} B_{\nu]}{}^c + b_5 e_{[\mu}{}^a e_{\nu]}{}^b B^{cd} B^{cd} \\ &\quad + b_6 \Omega_{[\mu}{}^{ab} B_{\nu]} + b_7 B_{[\mu}{}^{[a} f_{\nu]}{}^{b]} + b_8 B^{ab} f_{[\mu,\nu]} + b_9 e_{[\mu}{}^{[a} B^{b]c} f_{\nu]}{}^c + b_{10} f_{[\mu}{}^{[a} f_{\nu]}{}^{b]} \\ \Delta T_{\mu\nu}{}^a &= b_{11} \Omega_{[\mu}{}^{ab} f_{\nu]}{}^b + b_{12} f_{[\mu}{}^a B_{\nu]}\end{aligned}\quad (33)$$

The solution has the form (again using all possible field redefinitions):

$$\begin{aligned}b_2 &= -\frac{b_6}{4m}, & b_3 &= -\frac{b_6}{m}, & b_4 &= -\frac{2b_1}{(d-2)}, & b_5 &= 0 \\ b_7 &= -\frac{2mb_1}{(d-2)} - \frac{b_6}{2}, & b_8 &= -\frac{b_6}{2}, & b_9 &= -\frac{2mb_1}{(d-2)} \\ b_{10} &= -\frac{m^2 b_1}{(d-2)} - \frac{mb_6}{4}, & b_{11} &= 2b_1, & b_{12} &= \frac{4mb_1}{(d-2)} + b_6\end{aligned}$$

Gauge transformations for the deformed curvatures look like:

$$\begin{aligned}\delta \hat{R}_{\mu\nu}{}^{ab} &= 2b_1 \mathcal{F}_{\mu\nu}{}^c [{}^a \eta^{b]c} - b_6 \mathcal{B}_{\mu\nu} \eta^{ab} + b_7 \mathcal{B}_{[\mu,\nu]}{}^{[a} \xi^{b]} - b_8 \mathcal{B}_{[\mu}{}^{ab} \xi_{\nu]} - b_9 e_{[\mu}{}^{[a} \mathcal{B}_{\nu]}{}^{b]c} \xi^c \\ &\quad + 2b_{10} \mathcal{T}_{\mu\nu}{}^{[a} \xi^{b]} + b_6 \mathcal{F}_{\mu\nu}{}^{ab} \xi \\ \delta \hat{T}_{\mu\nu}{}^a &= -2b_1 \eta^{ab} \mathcal{T}_{\mu\nu}{}^b + 2b_1 \mathcal{F}_{\mu\nu}{}^{ab} \xi^b - b_{12} \mathcal{B}_{\mu\nu} \xi^a + b_{12} \mathcal{T}_{\mu\nu}{}^a \xi\end{aligned}\quad (34)$$

As for the partially massless curvatures deformations they again correspond to the minimal substitution rules:

$$\begin{aligned}\Delta \mathcal{F}_{\mu\nu}{}^{ab} &= -b_0 \omega_{[\mu}{}^c [{}^a \Omega_{\nu]}{}^{b]c} + \frac{mb_0}{(d-2)} [B_{[\mu}{}^{[a} h_{\nu]}{}^{b]} - e_{[\mu}{}^{[a} B^{b]c} h_{\nu]}{}^c] \\ \Delta \mathcal{T}_{\mu\nu}{}^a &= -b_0 \omega_{[\mu}{}^{ab} f_{\nu]}{}^b - b_0 \Omega_{[\mu}{}^{ab} h_{\nu]}{}^b - \frac{2mb_0}{(d-2)} h_{[\mu}{}^a B_{\nu]} \\ \Delta \mathcal{B}_{\mu}{}^{ab} &= -b_0 \omega_{\mu}{}^c [{}^a B^{b]c} \\ \Delta \mathcal{B}_{\mu\nu} &= b_0 B_{[\mu}{}^a h_{\nu]}{}^a + \frac{mb_0}{2} f_{[\mu}{}^a h_{\nu]}{}^a\end{aligned}\quad (35)$$

while their transformations have the form:

$$\begin{aligned}\delta \hat{\mathcal{F}}_{\mu\nu}{}^{ab} &= -b_0 \mathcal{F}_{\mu\nu}{}^c [{}^a \hat{\eta}^{b]c} - b_0 R_{\mu\nu}{}^c [{}^a \eta^{b]c} + \frac{mb_0}{(d-2)} [\mathcal{B}_{[\mu,\nu]}{}^{[a} \hat{\xi}^{b]} + e_{[\mu}{}^{[a} \mathcal{B}_{\nu]}{}^{b]c} \hat{\xi}^c] \\ \delta \hat{\mathcal{T}}_{\mu\nu}{}^a &= b_0 \hat{\eta}^{ab} \mathcal{T}_{\mu\nu}{}^b - b_0 \mathcal{F}_{\mu\nu}{}^{ab} \hat{\xi}^b + \frac{2mb_0}{(d-2)} \mathcal{B}_{\mu\nu} \hat{\xi}^a \\ &\quad + b_0 \eta^{ab} \mathcal{T}_{\mu\nu}{}^b - b_0 R_{\mu\nu}{}^{ab} \xi^b - \frac{2mb_0}{(d-2)} \mathcal{T}_{\mu\nu}{}^a \xi \\ \delta \hat{\mathcal{B}}_{\mu}{}^{ab} &= -b_0 \mathcal{B}_{\mu}{}^c [{}^a \hat{\eta}^{b]c} \\ \delta \hat{\mathcal{B}}_{\mu\nu} &= b_0 \mathcal{B}_{[\mu,\nu]}{}^a \hat{\xi}^a + \frac{mb_0}{2} \mathcal{F}_{\mu\nu}{}^a \hat{\xi}^a - \frac{mb_0}{2} \mathcal{T}_{\mu\nu}{}^a \xi^a\end{aligned}\quad (36)$$

Now let us consider the following Lagrangian:

$$\begin{aligned}\mathcal{L} = & a_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \hat{\mathcal{F}}_{\mu\nu}^{ab} \hat{\mathcal{F}}_{\alpha\beta}^{cd} + a_2 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \hat{\mathcal{B}}_{\mu}^{ac} \hat{\mathcal{B}}_{\nu}^{bc} + a_3 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \hat{\mathcal{B}}_{\mu}^{ab} \hat{\mathcal{T}}_{\nu\alpha}^c + a_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \hat{R}_{\mu\nu}^{ab} \hat{R}_{\alpha\beta}^{cd} \\ & + a_4 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ab} \mathcal{F}_{\alpha\beta}^{cd} h_{\gamma}^e + a_5 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{B}_{\mu}^{ad} \mathcal{B}_{\nu}^{bd} h_{\alpha}^c + a_6 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} R_{\mu\nu}^{ab} \mathcal{B}_{\alpha}^{cd} B_{\beta} \end{aligned} \quad (37)$$

Here the first line is the sum of the free Lagrangian for partially massless and massless spin-2 where initial curvatures are replaced by the deformed ones, while the second line contains possible abelian vertices. Note that one more possible term

$$\left\{ \begin{smallmatrix} \mu\nu\alpha\beta\gamma \\ abcde \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ab} R_{\alpha\beta}^{cd} f_{\gamma}^e$$

is on shell equivalent to some combination of others so we will not introduce it here.

Now let us require that this Lagrangian be gauge invariant. All $\hat{\eta}^{ab}$ variations vanish on-shell so we begin with $\hat{\xi}^a$ transformations. We have to take into account the part of variations that do not vanish on-shell, namely

$$\delta \hat{\mathcal{F}}_{\mu\nu}^{ab} = \frac{mb_0}{(d-2)} \mathcal{B}_{[\mu,\nu]}^{[a} \hat{\xi}^{b]}, \quad \delta \hat{\mathcal{T}}_{\mu\nu}^a = -b_0 \mathcal{F}_{\mu\nu}^{ab} \hat{\xi}^b$$

This produces:

$$\left[\frac{16m[(d-4)a_4 - a_1b_0]}{(d-2)} - ma_5 \right] \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ad} \mathcal{B}_{\alpha}^{bd} \hat{\xi}^c - a_3 b_0 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ad} \mathcal{B}_{\alpha}^{bc} \hat{\xi}^d$$

Using on-shell identity

$$0 \approx \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu,\alpha}^a \mathcal{B}_{\beta}^{bc} \hat{\xi}^d = \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ad} [-2\mathcal{B}_{\alpha}^{bd} \hat{\xi}^c + \mathcal{B}_{\alpha}^{bc} \hat{\xi}^d]$$

we obtain first equation:

$$\frac{16[(d-4)a_4 - a_1b_0]}{(d-2)} - a_5 + \frac{b_0}{2m^2} = 0 \quad (38)$$

For the η^{ab} transformations we have to take into account only

$$\delta \hat{R}_{\mu\nu}^{ab} = 2b_1 \mathcal{F}_{\mu\nu}^{c[a} \eta^{b]c}, \quad \delta \hat{\mathcal{F}}_{\mu\nu}^{ab} = -b_0 R_{\mu\nu}^{c[a} \eta^{b]c}$$

This gives us

$$4a_1 b_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ab} R_{\alpha\beta}^{ce} \eta^{de} - 8a_0 b_1 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \mathcal{F}_{\mu\nu}^{ae} \eta^{be} R_{\alpha\beta}^{cd}$$

using once again on-shell identity (17) we obtain

$$a_1 b_0 + 2a_0 b_1 = 0 \quad (39)$$

Non-vanishing on-shell part of the ξ^a transformations has the form

$$\delta \hat{R}_{\mu\nu}^{ab} = b_7 \mathcal{B}_{[\mu,\nu]}^{[a} \xi^{b]} - b_8 \mathcal{B}_{[\mu}^{ab} \xi_{\nu]}, \quad \delta \hat{\mathcal{T}}_{\mu\nu}^a = -b_0 R_{\mu\nu}^{ab} \xi^b$$

and we get

$$-(a_3b_0 + 8a_0b_8 - ma_6) \{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \} R_{\mu\nu}{}^{ad} \mathcal{B}_\alpha{}^{bc} \xi^d - 16a_0b_7 \{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \} R_{\mu\nu}{}^{ad} \mathcal{B}_\alpha{}^{bd} \xi^c$$

Using on-shell identity (31) (where $\mathcal{F}_{\mu\nu}{}^{ab}$ is replaced by $R_{\mu\nu}{}^{ab}$) we obtain

$$a_3b_0 + 8a_0b_8 - ma_6 + 8a_0b_7 = 0 \quad (40)$$

At last let us consider variations under ξ transformations:

$$\delta \hat{R}_{\mu\nu}{}^{ab} = b_6 \mathcal{F}_{\mu\nu}{}^{ab} \xi$$

This produces

$$[2a_0b_6 + \frac{ma_6}{2}] \{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \} \mathcal{F}_{\mu\nu}{}^{ab} R_{\alpha\beta}{}^{cd} \xi$$

and we obtain the last equation

$$4a_0b_6 + ma_6 = 0 \quad (41)$$

These equations have the following solution

$$b_1 = -\frac{b_0}{2}, \quad b_6 = 2mb_0, \quad a_5 = \frac{16(d-4)a_4}{(d-2)} - 16a_0b_0, \quad a_6 = -8a_0b_0 \quad (42)$$

Thus in general $d > 4$ case we have two independent vertices with parameters b_0 and a_4 ⁵. In $d = 4$ the parameter a_4 is absent leaving with one vertex only. Moreover we have explicitly checked that in this case all four derivative terms vanish on-shell. Note that, contrary to the self-interaction case, here our results do not agree with the one obtained in [22]. Table "2-2-2 couplings" in Appendix B of this paper gives four non-trivial vertices: two four derivatives ones and two vertices having no more than two derivatives. Moreover these results do not depend on space-time dimension. Due to very different approach used by authors of [22] it is not an easy task to see where and why such difference arises.

Conclusion

As we have seen application of Fradkin-Vasiliev formalism to the partially massless (and even more so in the massive) case appears to be more complicated and less elegant. The reason is that due to the large number of fields (main and Stueckelberg) and due to the presence of zero forms one faces a lot of ambiguities related with non-trivial on-shell identities and field redefinitions. Nevertheless, the formalism does work and allows one to obtain reasonable results.

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⁵As it will be shown in the Appendix in $d = 3$ case there exists one more cubic vertex with no more than two derivatives, but Fradkin-Vasiliev formalism we use here works in $d \geq 4$ dimensions only so we did not obtain such vertex here. Note also that in a frame-like gauge invariant formalism this vertex has been constructed in [20].

A Partially massless spin-2 in a constructive approach

In this appendix as an independent check for the results obtained in the main part we reconsider the same problems in the straightforward constructive approach.

A.1 Modified 1 and $\frac{1}{2}$ order formalism

In the constructive approach one usually assumes that the action can be considered as a row in the number of fields:

$$S = S_0 + S_1 + S_2 + \dots$$

where S_0 is free (quadratic) action, S_1 contains cubic terms, S_2 — quartic ones and so on. Similarly for the gauge transformations one assumes:

$$\delta\Phi = \delta_0\Phi + \delta_1\Phi + \delta_2\Phi + \dots$$

where δ_0 is non-homogeneous part, δ_1 is linear in fields and so on. Then variations of the action under any gauge transformations can also be represented as a row:

$$\delta S = \frac{\delta S_0}{\delta\Phi} \delta_0\Phi + \left(\frac{\delta S_1}{\delta\Phi} \delta_0\Phi + \frac{\delta S_0}{\delta\Phi} \delta_1\Phi \right) + \dots$$

First term simply implies that the free action S_0 must be gauge invariant under the initial gauge transformations $\delta_0\Phi$. Thus the first non-trivial level (that we will call linear approximation) looks as:

$$\frac{\delta S_1}{\delta\Phi} \delta_0\Phi + \frac{\delta S_0}{\delta\Phi} \delta_1\Phi = 0$$

Working with the frame-like formalism it is convenient to separate physical Φ and auxiliary Ω fields. Than in the honest first order formalism one has to achieve:

$$\frac{\delta S_1}{\delta\Phi} \delta_0\Phi + \frac{\delta S_1}{\delta\Omega} \delta_0\Omega + \frac{\delta S_0}{\delta\Phi} \delta_1\Phi + \frac{\delta S_0}{\delta\Omega} \delta_1\Omega = 0$$

It means that one has to consider the most general ansatz both the cubic vertex as well for the corrections to gauge transformations for the fields Φ and Ω . Taking into account that equations for auxiliary fields are algebraic and on their mass shell these fields are equivalent to the derivatives of physical ones, in supergravities there appeared a so-called 1 and $\frac{1}{2}$ order formalism. Schematically it looks like:

$$\left[\frac{\delta S_1}{\delta\Phi} \delta_0\Phi + \frac{\delta S_0}{\delta\Phi} \delta_1\Phi \right]_{\frac{\delta(S_0+S_1)}{\delta\Omega}=0} = 0$$

So one needs the most general ansatz for cubic vertex and physical fields gauge transformations only, but all calculations have to be done up to the terms proportional to the auxiliary fields equations, i.e. on their mass shell. Such approach turned out to be very effective, but it requires explicit solution of non-linear equations for auxiliary fields that can be rather complicated task. If we restrict ourselves with the linear approximation than there exists one more possibility that we will call modified 1 and $\frac{1}{2}$ order formalism. It looks like:

$$\left[\frac{\delta S_1}{\delta\Phi} \delta_0\Phi + \frac{\delta S_1}{\delta\Omega} \delta_0\Omega + \frac{\delta S_0}{\delta\Phi} \delta_1\Phi \right]_{\frac{\delta S_0}{\delta\Omega}=0} = 0$$

The main achievements here are twofold. At first, we have not consider the most general ansatz for cubic vertex but terms that are non-equivalent on auxiliary fields mass shell only. At second, we need explicit solution for the free auxiliary fields equations only. In what follows we will use such modified formalism.

A.2 Self-interaction

Our aim here is to determine the number of independent vertices so to simplify calculations in this and subsequent subsections we will heavily use all possible field redefinitions and all existing on-shell identities. We will work in a up-down approach i.e. we begin with four derivative terms, then we consider terms with three derivatives and so on.

Vertex 2 – 2 – 2 with four derivatives The only (on-shell non-trivial) possibility here is:

$$\mathcal{L}_{14a} = a_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_\mu \Omega_\nu^{ab} \Omega_\alpha^{ce} \Omega_\beta^{de}$$

Vertex 2 – 1 – 1 with four derivatives The most general⁶ ansatz looks like:

$$\begin{aligned} \mathcal{L}_{14b} = & \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} [a_1 \Omega_\mu^{ab} D_\nu B^{cd} B^{cd} + a_2 \Omega_\mu^{ac} D_\nu B^{bd} B^{cd} + a_3 \Omega_\mu^{ac} D_\nu B^{cd} B^{bd} \\ & + a_4 \Omega_\mu^{cd} D_\nu B^{ab} B^{cd} + a_5 \Omega_\mu^{cd} D_\nu B^{ac} B^{bd} + a_6 \Omega_\mu^{cd} D_\nu B^{cd} B^{ab}] \end{aligned}$$

But we have three possible field redefinitions:

$$\begin{aligned} f_\mu^a & \Rightarrow f_\mu^a + \kappa_1 B^{ab} B_\mu^a + \kappa_2 e_\mu^a B^2 \\ B_\mu & \Rightarrow B_\mu + \kappa_3 \Omega_\mu^{ab} B^{ab} \end{aligned}$$

and two on-shell identities (up to lower derivative terms):

$$\begin{aligned} 0 & \approx \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} D_\mu \Omega_{\nu,\alpha}^d B^{ab} B^{cd} = \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} D_\mu \Omega_\nu^{cd} [B^{ab} B^{cd} - 2B^{ac} B^{bd}] \\ & = \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \Omega_\mu^{cd} [D_\nu B^{ab} B^{cd} - 4D_\mu B^{ac} B^{bd} + D_\nu B^{cd} B^{ab}] \\ 0 & \approx \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} \Omega_\mu^{ab} D_\nu B_{\alpha\beta} B^{cd} = 2 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} [\Omega_\mu^{ab} B^{cd} - 4\Omega_\mu^{ac} B^{bd} + \Omega_\mu^{cd} B^{ab}] D_\nu B^{cd} \end{aligned}$$

Thus we have one independent vertex only in agreement with fact that there exists only one cubic 2 – 1 – 1 vertex with three derivatives for the massless fields. In what follows we will use

$$\mathcal{L}_{14b} = a_1 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} D_\mu \Omega_\nu^{cd} B^{ac} B^{bd}$$

Vertex 2 – 2 – 1 with three derivatives Here the most general ansatz has the form:

$$\begin{aligned} \mathcal{L}_{13} = & \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} [b_1 D_\mu \Omega_\nu^{ab} B^{cd} f_\alpha^d + b_2 D_\mu \Omega_\nu^{ad} B^{bd} f_\alpha^c + b_3 D_\mu \Omega_\nu^{ad} B^{bc} f_\alpha^d] \\ & + \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} [b_4 \Omega_\mu^{ab} \Omega_\nu^{cd} B^{cd} + b_5 \Omega_\mu^{ac} \Omega_\nu^{cd} B^{bd}] \end{aligned}$$

First of all note that in this case we have one possible field redefinition

$$f_\mu^a \Rightarrow f_\mu^a + \kappa_4 B^{ab} f_\mu^b$$

⁶Up to the terms that are equivalent on-shell

and one on-shell identity (again up to the lower derivative terms)

$$0 \approx \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_\mu \Omega_{\nu,\alpha}{}^a B^{bc} f_\beta{}^d = \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} D_\mu \Omega_\nu{}^{ad} [B^{bc} f_\alpha{}^d - 2B^{bd} f_\alpha{}^c]$$

Moreover, it is easy to check that invariance under the ξ^a transformations requires $b_2 = -2b_3$, so the terms with the coefficients $b_{2,3}$ vanish on-shell, while the one with coefficient b_1 can be removed by field redefinition.

Collecting all things together let us consider the following cubic Lagrangian:

$$\begin{aligned} \mathcal{L}_1 = & a_0 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_\mu \Omega_\nu{}^{ab} \Omega_\alpha{}^{ce} \Omega_\beta{}^{de} + a_1 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} D_\mu \Omega_\nu{}^{cd} B^{ac} B^{bd} \\ & + \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} [b_4 \Omega_\mu{}^{ab} \Omega_\nu{}^{cd} B^{cd} + b_5 \Omega_\mu{}^{ac} \Omega_\nu{}^{cd} B^{bd}] \end{aligned} \quad (43)$$

η^{ab} transformations produce the following variations for this Lagrangian:

$$\begin{aligned} \delta_0 \mathcal{L}_1 = & \frac{2m(d-4)a_0}{(d-2)} \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} [D_\mu \Omega_\nu{}^{ab} B^{cd} \eta^{cd} - 4D_\mu \Omega_\nu{}^{ac} B^{cd} \eta^{bd}] \\ & + \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} D_\mu \Omega_\nu{}^{cd} \left[\frac{2m(d-4)a_0}{(d-2)} B^{cd} \eta^{ab} - 2ma_1 B^{ac} \eta^{bd} \right] \\ & + \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} [b_4 D_\mu B^{cd} (\Omega_\nu{}^{ab} \eta^{cd} - \Omega_\nu{}^{cd} \eta^{ab}) + b_5 D_\mu B^{ac} (\Omega_\nu{}^{bd} \eta^{cd} - \Omega_\nu{}^{cd} \eta^{bd})] \\ & - m \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} [b_4 \Omega_\mu{}^{ab} \Omega_\nu{}^{cd} \eta^{cd} + b_5 \Omega_\mu{}^{ac} \Omega_\nu{}^{cd} \eta^{bd}] \end{aligned}$$

The terms in the first line can be compensated by the following corrections to gauge transformations:

$$\delta_1 f_\mu{}^a = \alpha_1 B^{ab} \eta_\mu{}^b + \alpha_2 \eta^{ab} B_\mu{}^b + \alpha_3 e_\mu{}^a (B\eta)$$

while for the second line we use on-shell identity

$$0 \approx D_\mu \Omega_{\nu,\alpha}{}^d B^{ad} \eta^{bc} = \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} D_\mu \Omega_\nu{}^{cd} [B^{cd} \eta^{ab} - 2B^{ac} \eta^{bd}]$$

and obtain

$$a_1 = \frac{2(d-4)a_0}{(d-2)}$$

The remaining terms cannot be compensated by any corrections to gauge transformations so we have to put

$$b_4 = b_5 = 0$$

Thus we get rather simple vertex with four derivatives. But such vertex exists in $d > 4$ dimensions only, while it is well known that in $d = 4$ there exists cubic vertex having no more than two derivatives. So we proceed and consider the following ansatz:

$$\begin{aligned} \mathcal{L}_1 = & c_1 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \Omega_\mu{}^{ad} \Omega_\nu{}^{bd} f_\alpha{}^c + c_2 \left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_\mu \Omega_\nu{}^{ab} f_\alpha{}^c f_\beta{}^d \\ & + c_3 e^\mu{}_a B^2 f_\mu{}^a + c_4 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} f_\mu{}^a B^{bc} D_\nu B_\alpha \\ & + d_1 \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \Omega_\mu{}^{ab} B_\nu f_\alpha{}^c + d_2 \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} f_\mu{}^a B^{bc} f_\nu{}^c \end{aligned} \quad (44)$$

ξ^a transformations produce the following variations:

$$\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} [-(2c_1 + 4c_2) D_\mu \Omega_\nu{}^{ad} \Omega_\alpha{}^{bd} \xi^c - 2c_2 D_\mu \Omega_\nu{}^{ab} \Omega_\alpha{}^{cd} \xi^d] - \left(\frac{4m(d-3)c_2}{(d-2)} + d_1 \right) \left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} D_\mu \Omega_\nu{}^{ab} B_\alpha \xi^c$$

$$\begin{aligned}
& +e^{\mu a}[-(2c_3+c_4)B^{bc}D_\mu B^{bc}\xi^a-2c_4D_\mu B^{ab}B^{bc}\xi^c]+(d_2+mc_4)\{\overset{\mu\nu}{ab}\}D_\mu B^{ac}[f_\nu{}^c\xi^b-f_\nu{}^b\xi^c] \\
& +e^\mu{}_a[(d_1-d_2)\Omega_\mu{}^{bc}B^{bc}\xi^a+(mc_4-d_2)B^{ab}\Omega_\mu{}^{bc}\xi^c+(2d_1-d_2-mc_4)\Omega_\mu{}^{ab}B^{bc}\xi^c] \\
& +(\frac{4m^2(d-3)c_2}{(d-2)}+md_1)\{\overset{\mu\nu}{ab}\}\Omega_\mu{}^{ac}[f_\nu{}^b\xi^c-f_\nu{}^c\xi^b]-2m(d_2+mc_4)e^\mu{}_aB^{ab}\xi^bB_\mu
\end{aligned}$$

If we put

$$c_1 = -2c_2, \quad c_3 = -\frac{c_4}{2}, \quad d_1 = d_2 = -mc_4 \quad (45)$$

we obtain

$$\begin{aligned}
& -c_2\{\overset{\mu\nu\alpha}{abc}\}\mathcal{F}_{\mu\nu}{}^{ab}\Omega_\alpha{}^{cd}\xi^d-2c_4e^\mu{}_a\mathcal{B}_\mu{}^{ab}B^{bc}\xi^c+2m(c_4-2c_2)e^\mu{}_aB^{ab}\Omega_\mu{}^{bc}\xi^c \\
& -(\frac{4m(d-3)c_2}{(d-2)}-mc_4)\{\overset{\mu\nu\alpha}{abc}\}D_\mu\Omega_\nu{}^{ab}B_\alpha\xi^c+(\frac{4m^2(d-3)c_2}{(d-2)}-m^2c_4)\{\overset{\mu\nu}{ab}\}\Omega_\mu{}^{ac}[f_\nu{}^b\xi^c-f_\nu{}^c\xi^b]
\end{aligned}$$

First two terms can be compensated by the following corrections to gauge transformations:

$$\delta_1 f_\mu{}^a \sim \Omega_\mu{}^{ab}\xi^b, \quad \delta_1 B_\mu \sim B^{ab}\xi^b$$

while the remaining terms require

$$c_4 = 2c_2, \quad c_4 = \frac{4(d-3)c_2}{(d-2)} \Leftrightarrow d = 4$$

thus such solution indeed exists in $d = 4$ dimensions only.

η^{ab} **transformations** With the same restrictions on parameters we get

$$\begin{aligned}
& c_2\{\overset{\mu\nu\alpha}{abc}\}\mathcal{F}_{\mu\nu}{}^{ab}\eta^{cd}f_\alpha{}^d-4m(c_4-2c_2)\{\overset{\mu\nu}{ab}\}\Omega_\mu{}^{ac}\eta^{bc}B_\nu \\
& -2m(c_4-2c_2)e^\mu{}_aB^{ab}\eta^{bc}f_\mu{}^c-m^2(c_4-\frac{4(d-3)c_2}{(d-2)})\{\overset{\mu\nu}{ab}\}f_\mu{}^a\eta^{bc}f_\nu{}^c
\end{aligned}$$

The first term can be compensated by the following correction

$$\delta f_\mu{}^a \sim \eta^{ab}f_\mu{}^b$$

while the remaining ones again give

$$c_4 = 2c_2, \quad d = 4$$

ξ **transformations** Similarly:

$$m(-c_4+\frac{4(d-3)c_2}{(d-2)})\{\overset{\mu\nu\alpha}{abc}\}D_\mu\Omega_\nu{}^{ab}f_\alpha{}^c+2m(-2c_2+c_4)\{\overset{\mu\nu}{ab}\}\Omega_\mu{}^{ac}\Omega_\nu{}^{bc}+\frac{m(d-4)c_4}{(d-2)}B^2$$

in agreement with all previous results.

A.3 Gravitational interaction

In this case we have to consider variations for all five transformations: $\hat{\eta}^{ab}$, $\hat{\xi}^a$ for graviton and η^{ab} , ξ^a , ξ for partially massless spin-2. It requires rather long calculations so we will not reproduce it here restricting ourselves with the main results.

Vertices with four derivatives Using on-shell identities and field redefinitions can be written as follows

$$\mathcal{L}_{14} = a_0 \left\{ \frac{\mu\nu\alpha\beta}{abcd} \right\} [2D_\mu \Omega_\nu^{ab} \Omega_\alpha^{ce} \omega_\beta^{de} + D_\mu \omega_\nu^{ab} \Omega_\alpha^{ce} \Omega_\beta^{de}] + a_1 \left\{ \frac{\mu\nu}{ab} \right\} D_\mu \omega_\nu^{cd} B^{ac} B^{bd}$$

Vertex with three derivatives As in the case of self-interaction all terms of the form $D\Omega B h$ and $D\omega B f$ vanish on-shell or can be removed by field redefinitions. This leaves us with:

$$\mathcal{L}_{13} = \left\{ \frac{\mu\nu}{ab} \right\} [b_1 \Omega_\mu^{ab} B^{cd} \omega_\nu^{cd} + b_2 \Omega_\mu^{ac} B^{cd} \omega_\nu^{bd} + b_3 \Omega_\mu^{cd} B^{cd} \omega_\nu^{ab} + b_4 \Omega_\mu^{cd} B^{ab} \omega_\nu^{cd}]$$

Note that this structure is similar to one of $2-2-1$ vertex with three derivatives that plays important role in the electromagnetic interactions for spin 2 particles [17].

Variations of order m require⁷:

$$b_2 = -4b_1, \quad b_3 = b_1, \quad b_1 + b_4 = -\frac{2m(d-4)a_0}{(d-2)}, \quad a_1 = \frac{4(d-4)a_0}{(d-2)}$$

provided we introduce the following corrections to the gauge transformations:

$$\delta B_\mu = b_4 [\omega_\mu^{ab} \eta^{ab} - \Omega_\mu^{ab} \hat{\eta}^{ab}]$$

Thus at this stage we have two independent parameters a_0 and say b_4 .

Vertices with two derivatives The most general on-shell non-equivalent form looks like

$$\begin{aligned} \mathcal{L}_{12} = & \left\{ \frac{\mu\nu\alpha}{abc} \right\} [c_1 \Omega_\mu^{ad} \Omega_\nu^{bd} h_\alpha^c + c_2 \Omega_\mu^{ab} \Omega_\nu^{cd} h_\alpha^c] \\ & + \left\{ \frac{\mu\nu\alpha}{abc} \right\} [c_3 \Omega_\mu^{ad} \omega_\nu^{bd} f_\alpha^c + c_4 \Omega_\mu^{ab} \omega_\nu^{cd} f_\alpha^d + c_5 \Omega_\mu^{ad} \omega_\nu^{bc} f_\alpha^d] \\ & + e^\mu_a [c_6 B^2 h_\mu^a + c_7 B^{ab} B^{bc} h_\mu^c] \end{aligned} \quad (46)$$

Variations of order m^2 require

$$\begin{aligned} c_1 + c_5 &= -2mb_1, & c_2 + c_5 &= m(b_4 - b_1) \\ c_3 &= -2c_5, & c_4 &= -c_4, & c_7 &= 4c_6 - \frac{2ma_1}{(d-2)} \end{aligned}$$

while corresponding corrections to the gauge transformations gave the form:

$$\begin{aligned} \delta f_\mu^a &= -2c_5 (\hat{\eta}^{ab} f_\mu^b - \omega_\mu^{ab} \xi^b) + 2(c_1 - \frac{4m^2(d-4)a_0}{(d-2)}) (\eta^{ab} h_\mu^b - \Omega_\mu^{ab} \hat{\xi}^b) \\ \delta h_\mu^a &= 2c_5 (\eta^{ab} f_\mu^b - \Omega_\mu^{ab} \xi^b), & \delta B_\mu &= 2c_6 B_\mu^a \hat{\xi}^a \end{aligned} \quad (47)$$

⁷We organize variations by the dimensionality of coefficients. E.g. variations of order m means coefficients of the form ma or b and so on.

Vertex with one derivative The most general ansatz is:

$$\mathcal{L}_{11} = \{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \} [d_1 \omega_\mu^{ab} f_\nu^c B_\alpha + d_2 \Omega_\mu^{ab} h_\nu^c B_\alpha] + d_3 \{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \} B^{ab} f_\mu^c h_\nu^c$$

Note that the only possible term without derivatives

$$\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \} f_\mu^a f_\nu^b h_\alpha^c$$

is forbidden by the invariance under the ξ transformations.

First of all note that solution with non-zero parameter c_5 exists in $d = 3$ dimensions only. Recall that the Fradkin-Vasiliev formalism we use in the main part works in $d \geq 4$ dimensions only so it cannot reproduce such vertex. Note however that in the frame-like gauge invariant formalism this vertex (having no more than two derivatives) has been constructed in [20]. For the general $d > 3$ case we obtain two independent solutions with a_0 and b_4 as free parameters:

$$c_1 = -2mb_1, \quad c_2 = m(b_4 - b_1), \quad c_6 = \frac{m(d-3)b_4}{(d-2)}$$

$$c_3 = c_4 = c_5 = d_1 = d_2 = d_3 = 0$$

Note at last that in $d = 4$ dimensions parameter a_0 is absent leaving us with one vertex only, moreover in this case all four derivative terms vanish on-shell.

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